An approach to **innocent strategies as graphs**

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Denotational semantics

$$\downarrow$$

$\approx$ (game semantics / ludics)

$$\uparrow$$

Syntax

"the best of both worlds"

$$\begin{array}{ccc} \text{proofs} & \longrightarrow & \text{(tree) strategies} \\ \downarrow & & \downarrow \\ \text{proof nets} & \longrightarrow & \textbf{graph strategies} \end{array}$$

Ludics 1/6

ludics = **focalized**, **untyped** version of MALL sequent calculus.

(polarized, focalized) MALL sequent calculus proofs $\longrightarrow$ **designs**

(MALL = Multiplicative-Additive Linear Logic)

Ludics 2/6

$$\dfrac{\dfrac{\vdash P_1, \Gamma_1 \quad \vdash P_2, \Gamma_1}{\vdash P_1 \& P_2, \Gamma_1} \qquad \vdash N, \Gamma_2}{\vdash (P_1 \& P_2) \otimes N, \Gamma_1, \Gamma_2}$$

$$\downarrow$$

additive rule (immediate conflict)

$$\dfrac{(\xi 1, \{1\})^- \quad (\xi 1, \{2\})^- \qquad \ldots \quad (\xi 2, J)^-}{(\xi, \{1, 2\})^+}$$

Ludics 3/6

- Formulas organized by alternating **clusters** of **positive** ($\otimes$, $\oplus$) (resp. **negative** ($\parr$, $\&$)) formulas
- Each cluster becomes an **address** (cf. type $\mathtt{bool}_2 \longrightarrow \mathtt{bool}_1 \longrightarrow \mathtt{bool}_\epsilon$ in game semantics)

Ludics 4/6

- Logical rules expressed in terms of **actions** $= (\xi, I)$ (I finite set of relative immediate subaddresses of ξ)

- We say that (ξ, I) generates ξi ($i \in I$)

- A negative rule involving & gives rise to actions $(\xi, I_1), \ldots, (\xi, I_n)$ on the *same* address

Ludics 5/6

Looks exotic? Just (sorts of) Böhm trees:

• negative $\xi \to \lambda x_1 \ldots x_i \ldots x_n.P$

• positive $\xi i \to x_i M_1 \ldots M_p$ where x_i is bound higher up at a negative node of address ξ

Ludics 6/6

Full syntax for Girard's designs (Curien 2001):

$$M ::= \{J = \lambda\{x_j : \ j \in J\}.P_J : \ J \in \mathcal{P}_f(\omega)\}$$
$$P ::= (x \cdot I)\{M_i : \ i \in I\} \mid \Omega \mid \maltese$$

L-nets 1/6

An *L-net* (**Faggian/Maurel**) $\mathfrak{D}$ is given by:

• An interface $\vdash \Lambda$ (positive) or $\xi \vdash \Lambda$ (negative).

• A set A of nodes (or events) which are *labelled by polarized actions* (notation $k = (\xi, I)$)

• A structure on A of *directed acyclic bipartite graph* (if $k \leftarrow k'$, the two nodes have opposite polarity) which satisfies (for all k):

L-nets 2/6

- *Views.* All the addresses used in $k^{\downarrow} = \{k', k' \overset{+}{\leftarrow} k\}$ are distinct.

- *Parents.* If $k = (\sigma, I)$, then either $\sigma \in$ interface (with same polarity), or it has been generated by (the action of) a $c \overset{+}{\leftarrow} k$ (of opposite polarity). Moreover, if k is negative, and $b \leftarrow k$, then $b = c$ (*innocence*!)

- *Positivity.* (k maximal w.r.t. $\overset{+}{\leftarrow}$) $\Rightarrow$ (k positive)

L-nets 3/6

- *Sibling.* Two nodes in an additive pair have distinct labels (in the example above, $\{1\} \neq \{2\}$).
- *Additives.* If $k_1 = (\xi, K_1)$, $k_2 = (\xi, K_2)$, $\exists w_1, w_2$ in the same additive rule such that $w_1 \stackrel{+}{\leftarrow} k_1$, and $w_2 \stackrel{+}{\leftarrow} k_2$.

("two events on the same address are in conflict")

(So far = L-nets, one more condition for L_S-nets)

L-nets 4/6

Fact. For each pair of distinct nodes k, k' of an L-net $\mathfrak{D}$, the sets of actions of $k^{\downarrow}$ and $k'^{\downarrow}$ are different.

$\longrightarrow$ L-nets as sets of (positive) **views** (= L-nets with a maximal element, and whose nodes **are** actions).

Very useful for *superpositions as mere unions*.

(cf. event structures presented as configuration structures)

L-nets 5/6

A *switching edge* of a negative rule R has its target is in R.

A *switching path* uses at most one switching edge for each negative rule.

- *Cycles.* For all non-empty union C of switching cycles, there is an additive rule W not intersecting C, and a pair $w_1, w_2 \in W$ such that for some nodes $c_1, c_2 \in C$, $w_1 \stackrel{+}{\leftarrow} c_1$, and $w_2 \stackrel{+}{\leftarrow} c_2$.

L-nets 6/6

The condition *Cycles* is an anologue of Hughes and Van Glabbeek's *toggling* condition.

It is the key to sequentialization:

$\boxed{\text{every } L_S\text{-net has a splitting conclusion}}$

A gradient of sequentiality 1/5

- *L-forests.* Maximally sequential L-nets are *forests* (Girard's designs with *mix*).

- *parallel L-nets.* Minimally sequential L-nets = our notion of multiplicative-additive (untyped, focalized) proof-nets

A gradient of sequentiality 2/5

Algebraic presentation of parallel L-nets:

$$
\boxed{
\begin{array}{lcl}
\mathfrak{D} & := & \mathfrak{D}^{+} \mid \mathfrak{D}^{-}_{\sigma} \\
\mathfrak{D}^{+} & := & \biguplus \mathfrak{E}^{+} \\
\mathfrak{E}^{+} & := & k^{+} \mid \bigcup \ (\xi, I)^{+} \circ \mathfrak{D}^{-}_{\xi_i} \\
\mathfrak{D}^{-}_{\sigma} & := & \bigcup_{add} \quad (\sigma, J)^{-} \circ \mathfrak{D}^{+}
\end{array}
}
$$

A gradient of sequentiality 3/5

- *Rooting.* $x \circ \mathfrak{D}^+$: the node x is added, and only edges enforced by condition *Parents* are added.
- *Boxing.* $x \cdot \mathfrak{D}^+$: the node x is added *below all the conclusions of* $\mathfrak{D}$.
- *Additive union.* $\cup_{add} \quad \mathfrak{D}_I$: selective union (only the views which are common to all $\mathfrak{D}_I$'s are shared)

(and associated destructors)

A gradient of sequentiality 4/5

Algebraic presentation of L-forests:

$$\boxed{\begin{array}{lcl} \mathfrak{D} & := & \mathfrak{D}^{+} \mid \mathfrak{D}^{-}_{\sigma} \\ \mathfrak{D}^{+} & := & \uplus\, \mathfrak{E}^{+} \\ \mathfrak{E}^{+} & := & k^{+} \mid \cup\ (\xi, I)^{+} \circ \mathfrak{D}^{-}_{\xi_i} \\ \mathfrak{D}^{-}_{\sigma} & := & \cup\ (\sigma, J)^{-}.\mathfrak{D}^{+} \end{array}}$$

A gradient of sequentiality 5/5

• Every L$_S$-net can be (non-deterministically) **sequentialized** to an L-forest.

• Every L-forest (more generally, every L$_S$-net) with leaves decorated by sets of actions ("axioms") can be **desequentialized** to a parallel L-net.

The two procedures can be applied so as to be **inverse** to each other.

Further work

• Characterization of minimal sequentiality and of the induced equational theory on L-forests

• sequentialization/desequentialization à la carte: Di Giambernardino-Faggian (multiplicative)

• What kind of proof nets do we get when restoring types?

A wider picture

Aim: to link proof theory, game semantics, and concurrency theory.

L-nets $\rightarrow$ (typed) event structures

L-net normalisation (Faggian-Maurel) $\rightarrow$ parallel composition (+ synchronization) of (typed) event structures (Faggian-Piccolo) (cf. Varacca-Yoshida)

Operations on L-nets $\leftrightarrow$ operations on event structures.